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K_2 of finite abelian group algebras

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ABSTRACT

In this paper, we represent $K_2(F[G \times Z_p])$ as the direct sum of $K_2(FG)$ and an elementary abelian p-group; using this we calculate $K_2(FG)$ when F is a finite field of odd prime characteristic and G is a finite abelian group of p^2-rank ≤ 1. We also compute $H^1_{DR}(FG)$ if F is a finite field of odd characteristic p and G is a finite abelian p-group.

1. Introduction

Let F be a finite field of characteristic p and G a finite abelian group. A general formula for $K_2(FG)$ has been given in [7], Theorem 6.7; it is the quotient of $G \otimes AG$ with A the unramified p-ring such that $F \approx A/p$. An upper bound for the order of $K_2(FG)$ was also given in [7]. However, it is not easy to determine the structure of $K_2(FG)$ directly from this quotient, even its precise order. The result of Dennis and Stein in [6] (Corollary 4.4(a)) implies that for a cyclic group C_n, $K_2(FC_n) = 1$. For an elementary abelian p-group G, Dennis, Keating and Stein [1] proved that $K_2(FG)$ is an elementary abelian p-group whose precise rank is also given. In Magurn [2], when F is a characteristic 2, $K_2(F[G \times \mathbb{Z}_2])$ is isomorphic to the direct sum of $K_2(FG)$ and an elementary abelian 2-group whose rank is determined. Using this result, Magurn calculated $K_2(FG)$ when G is a finite abelian group with 4-rank ≤ 1 and F is of characteristic 2. In Section 3 of this paper, we extend Magurn's results to the case when p is an odd prime. In Section 4, it will be shown that to get the precise order of $K_2(FG)$, the only thing we need to know is the orders of kernels of $\Omega^1_{FG/\mathbb{Z}} \to K_2(FG[t]/(t^k), (t))$, $k \equiv 1 \bmod p$, $k > p$, where F is of odd characteristic p, G is a finite abelian p-group. We determine the de Rham cohomology group $H^1_{DR}(FG)$ for arbitrary abelian p-groups G, and show how this cohomology group can be used to compute the above kernels in case G is an elementary abelian p-group.

2. Preliminaries

Suppose k is a commutative ring, A is a k-algebra, for an A-module M, a k-derivation from A to M is a k-linear map d: $A \to M$ such that

$$d(ab) = (da)b + a(db), \quad (a, b \in A).$$

The set of all such derivations $\mathrm{Der}_k(A, M)$ is an A-module, which is functorial in M. A universal k-derivation $d : A \to \Omega^1_{A/k}$ is defined by taking $\Omega^1_{A/k}$ to be the A-module defined by generators da $(a \in A)$, and relations

$$d(ab) = adb + bda, \qquad d(a + b) = da + dba, \quad b \in A, \text{ as well as } dc = 0, c \in k.$$

$\Omega^1_{A/K}$ is called Kähler differentials of A over k and the universality is expressed by the natural isomorphism $\mathrm{Hom}_{A\text{-mod}}(\Omega^1_{A/K}, M) \to \mathrm{Der}_k(A, M)$ sending f to $f \circ d$.

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The algebra of differential forms over an algebra A is the exterior algebra $\Omega^*_{A/K} = \bigoplus_q \Omega^q_{A/K}$, $\Omega^q_{A/K} = \wedge^q_A \Omega^1_{A/K}$, any element of Ω^q_A is called a differential form of degree q. The morphism $d : \Omega^*_{A/K} \to \Omega^*_{A/K}$ of degree $+1$, defined by $d(a_0da_1\cdots da_q) = da_0da_1\cdots da_q$, changes $(\Omega^*_{A/K}, d)$ into a complex. $(\Omega^*_{A/K}, d)$ is called the de Rham complex of A and the cohomology algebra $H^*_{DR}(A)$ is called the de Rham cohomology of A over k.

Let R be a commutative ring with identity. $\Phi_i(R)$ $(i \geq 2)$ is defined by the following exact sequence in [1],

$$1 \to \Phi_i(R) \to K_2(R[t]/(t^i)) \to K_2(R[t]/(t^{i-1})) \to 1.$$

The following theorem is due to Bloch [5].

Theorem 2.1. *If R is a commutative local F_p-algebra and p is odd, then*

$$\Phi_i(R) \approx \begin{cases} \Omega^1_{R/\mathbb{Z}} & i \not\equiv 0, 1 \bmod p, \\ \Omega^1_{R/\mathbb{Z}} \oplus R/R^{p^r} & i = mp^r, (p, m) = 1, r \geqq 1. \end{cases}$$

Note that there are no formulas for $\Phi_k(R)$ when $k \equiv 1 \bmod p$. In fact it is very difficult to determine them.

The following result of Magurn [2] gives the structure of $\Omega^1_{FG/Z}$.

Theorem 2.2. *Suppose F is a finite field of characteristic p, G is a finite abelian group, and $\bar{a}_1, \ldots, \bar{a}_r$ is an F_p-basis of G/G^p, then $\Omega^1_{FG/Z}$ is a free FG-module with basis $da_1, \ldots, da_r$.*

When R is a commutative ring with identity and I is a radical ideal, $K_2(R, I)$ is the abelian group which has a presentation with generators the Dennis–Stein symbols $\langle a, b\rangle$ for every $(a, b) \in R \times I \cup I \times R$ and the following relations

$(D1)$ $\langle a, b\rangle = -\langle b, a\rangle$ if $a \in I$;

$(D2)$ $\langle a, b\rangle + \langle a, c\rangle = \langle a, b + c - abc\rangle$ if $a \in I$ or $b, c \in I$;

$(D3)$ $\langle a, bc\rangle = \langle ab, c\rangle + \langle ac, b\rangle$ if $a \in I$.

Let $\widetilde{R}$ be a ring containing R. If $a \in I$ and $b \in R \cap \widetilde{R}^*$, then the image of $\langle a, b\rangle$ under the map $K_2(R, I) \to K_2(\widetilde{R})$ is the Steinberg symbol $\{1 - ab, b\}$; One can consult [3] to see more about Dennis–Stein symbols.

3. Adding Z_p summands to G

Suppose F is a finite field of characteristic p, G is a finite abelian group, and $Z_{p^r} = \langle\sigma\rangle$ is a cyclic group of order p^r. Let $A = F[G \times Z_{p^r}]$. Then there is a partial augmentation map $\varepsilon : A \to F[G]$ sending σ to 1; the kernel of ε is $I = (1-\sigma)A$. Since ε is a split surjective map and K_n are functors, we have a split exact sequence which is just a part of the long exact sequence in K-theory with respect to the pair (A, I):

$$1 \to K_2(A, I) \to K_2(A) \to K_2(FG) \to 1.$$

So obviously $K_2(A) \approx K_2(FG) \oplus K_2(A, I)$. From the isomorphisms

$$A \approx FG[t]/(t^{p^r} - 1) \approx FG[t]/(t^{p^r}),$$

it follows that $K_2(A, I) \approx K_2(FG[t]/(t^{p^r}), (t))$; thus

$$K_2(F[G \times Z_{p^r}]) \approx K_2(FG) \oplus K_2(FG[t]/(t^{p^r}), (t)). \tag{3.1}$$

Theorem 3.1. *Suppose F is a finite field of characteristic p, G is a finite abelian group whose Sylow p-subgroups is G_p, then $K_2(FG)$ is a finite p-group annihilated by the exponent of G_p.*

Proof. Decompose G as the direct sum $G_p \oplus H$ with G_P the Sylow p-subgroup of G. BY Maschke's Theorem, FH is a semisimple ring. Then the Wedderburn–Artin Theorem implies that $FH \approx \bigoplus_i F_i$, where F_i is a finite field with the same characteristic as F. Now $FG \approx (FH)[G_p] \approx \bigoplus_i F_iG_p$ and $K_2(FG) \approx \bigoplus_i K_2(F_iG_p)$.

Decompose G_p as a finite direct sum of cyclic p-groups. $G_p \approx C_{p^{l_1}} \oplus \cdots \oplus C_{p^{l_r}}$, where $p^{l_1} \leq \cdots \leq p^{l_r}$. Let $R_{ij} = F_i[C_p^{l_1} \oplus \cdots \oplus C_p^{l_j}]$, $1 \leq j \leq r$. Using (3.1) we get the following isomorphism

$$K_2(F_iG_p) \approx \bigoplus_{j=1}^{r-1} K_2(R_{ij}[t]/(t^{p^{l_{j+1}}}), (t)) \bigoplus K_2(F_iC_{p^{l_1}}).$$

The last summand vanishes; and $K_2(R_{ij}[t]/(t^{p^{l_{j+1}}}), (t))$ has exponent $p^{l_{j+1}}$ since it is generated by $\langle a, b\rangle$ with $a \in (t)$, and $p^{l_{j+1}}\langle a, b\rangle = \langle a^{p^{l_{j+1}}} b^{p^{l_{j+1}}-1}, b\rangle = \langle 0, b\rangle = 0$ since $a^{p^{l_{j+1}}} \in (t^{p^{l_{j+1}}})$. Thus $K_2(F_iG_p)$ has exponent p^{l_r} and accordingly $K_2(FG)$ is a p-group with exponent p^{l_r}. $\square$

When FG is not a local F_p-algebra, one cannot use Theorem 2.5 directly to compute $\Phi_i(FG)$, In order to be more convenient in concrete calculation, we partially extend Theorem 2.1 to the form we need.

Theorem 3.2. *Suppose F is a finite field of odd prime characteristic and G is a finite abelian group and $R = FG$. Then Bloch's calculation still works for FG:*

$$\Phi_i(FG) \approx \begin{cases} \Omega^1_{FG/\mathbb{Z}} & i \not\equiv 0, 1 \bmod p, \\ \Omega^1_{FG/\mathbb{Z}} \oplus R/R^{p^r} & i = mp^r, (p, m) = 1, r \geqq 1. \end{cases}$$

Proof. As in the proof of Theorem 3.1, $FG \approx \bigoplus_j F_jG_p$, where G_p is the Sylow p-subgroup of G and F_j has the same characteristic as F. Say $G_p \approx C_{p^{l_1}} \oplus \cdots \oplus C_{p^{l_r}} = \langle g_1 \rangle \times \cdots \times \langle g_r \rangle$. By the definition of Φ_i we have $\Phi_i(FG) \approx \bigoplus_j \Phi_i(F_jG_p)$. Since F_jG_p is a local F_p-algebra, then by Theorem 2.1

$$\Phi_i(F_jG_p) \approx \begin{cases} \Omega^1_{F_jG_p/\mathbb{Z}} & i \not\equiv 0, 1 \bmod p, \\ \Omega^1_{F_jG_p/\mathbb{Z}} \oplus F_jG_p/(F_jG_p)^{p^r} & i = mp^r (p, m) = 1, r \geqq 1. \end{cases}$$

By Theorem 2.2, $\Omega^1_{FG/\mathbb{Z}}$ is a free FG-module with basis $dg_1, \ldots, dg_r$, $\Omega^1_{F_jG_p/\mathbb{Z}}$ is a free F_jG_p-module with basis $dg_1, \ldots, dg_r$, so $\Omega^1_{FG/\mathbb{Z}} \approx \bigoplus_j \Omega^1_{F_jG_p/\mathbb{Z}}$ as abelian groups. Obviously $FG/(FG)^p \approx \bigoplus_j (F_jG_p)/(F_jG_p)^{p^r}$, the theorem now follows. □

The following theorem and corollary extend the results of Magurn [2] (Theorems 4 and 5) to the case when p is an odd prime.

Theorem 3.3. *Suppose F is finite field with p^f elements, p is an odd prime, G is a finite abelian group of order n, and r is the dimension of the F_p-space G/G_p. Then*

$$K_2(F[G \times \mathbb{Z}_p]) \approx K_2(FG) \oplus \mathbb{Z}_p^{fn(1-\frac{1}{p^r}+(p-1)r)}.$$

Proof. By (3.1) we have the following isomorphism

$$K_2(F[G \times \mathbb{Z}_p]) \approx K_2(FG) \oplus K_2(FG[t]/(t^p), (t)).$$

Since $p\langle a, b\rangle = \langle a^p b^{p-1}, b\rangle$, $K_2(FG[t]/(t^p), (t))$ is an elementary abelian p-group. By the isomorphism $F[G \times \mathbb{Z}_p] \approx FG[\mathbb{Z}_p] \approx FG[t]/(t^p)$ and the exact sequence

$$1 \to \Phi_i(FG) \to K_2(FG[t]/(t^i)) \to K_2(FG[t]/(t^{i-1})) \to 1, \quad 2 \le i \le p,$$

we have

$$|K_2(FG[t]/(t^p), (t))| = \prod_{i=2}^{p} |\Phi_i(FG)|.$$

By Theorem 3.2, $\Phi_i(FG) \approx \Omega^1_{FG/\mathbb{Z}}$, $2 \le i < p$, $\Phi_p(FG) \approx \Omega^1_{FG/\mathbb{Z}} \oplus FG/(FG)^p$. $\Omega^1_{FG/\mathbb{Z}}$ is a free FG-module of rank r, so it has rank nfr as an F_p-vector space. The group G has p^r pth power classes, so there are $n(1 - \frac{1}{p^r})$ elements of G that are not elements of G^p, hence $FG/F[G^p]$ has F_p-dimension $nf(1 - \frac{1}{p^r})$. Thus

$$\dim_{F_p}(K_2(FG[t]/(t^p), (t))) = (p-1)nfr + nf\left(1 - \frac{1}{p^r}\right) = nf\left(1 - \frac{1}{p^r} + (p-1)r\right).$$

The theorem now follows. □

Corollary 3.4. *Suppose F is a finite field with p^f elements, and G is a finite abelian group of order n with p-rank t, p^2-rank ≤ 1, then*

$$K_2(FG) \approx \mathbb{Z}_p^{nf(t-1)(p^t-1)/p^t}.$$

Proof. If $p = 2$, the theorem is just Theorem 5 in [2]; If p is an odd prime, repeated use of Theorem 3.3 yields the result. Since the process has been shown in Theorem 5 in [2], we omit the details. □

Example 1. A direct use of Theorem 3.4 yields $K_2(F_5[\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_{25} \times \mathbb{Z}_6]) \approx \mathbb{Z}_5^{7440}$.

4. The order of $K_2(F_qG)$

Suppose F is a finite field of odd prime characteristic p and G is a finite abelian group. By the definition of $\Phi_i(R)$ and the isomorphism $F[G \times \mathbb{Z}_{p^s}] \approx FG[t]/(t^{p^s})$, we have

$$|K_2(F[G \times \mathbb{Z}_{p^s}])| = |K_2(FG)| \cdot \prod_{i=2}^{p^s} |\Phi_i(FG)|.$$

When $i \not\equiv 1 \mod p$, the order of $\Phi_i(FG)$ can be determined by Theorem 3.2. When $i \equiv 1 \mod p$, it is very difficult to determine the precise order of $\Phi_i(FG)$. We have the following commutative diagram, let $R = FG$.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_2(R[t]/(t^{mp^r+1}),(t)) & \xrightarrow{f} & K_2(R[t]/(t^{mp^r+1})) & \longrightarrow & K_2(R) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & K_2(R[t]/(t^{mp^r}),(t)) & \longrightarrow & K_2(R[t]/(t^{mp^r})) & \longrightarrow & K_2(R) & \longrightarrow & 1. \end{array}$$

We will use the injectivity of f to determine whether a Dennis–Stein symbol is trivial in $K_2(R[t]/(t^{mp^r+1}),(t))$. By (1.6) and (1.10) in [3], we have the following exact sequences

$$\Omega^1_{R/\mathbb{Z}} \xrightarrow{\varphi_{m,r}} K_2(R[t]/(t^{mp^r+1}),(t)) \to K_2(R[t]/(t^{mp^r}),(t)) \to 1,$$

where $\varphi_{m,r}(adb) = \langle at^{mp^r}, b\rangle$. So we have $\Phi_{mp^r+1}(FG) \approx \Omega^1_{FG/\mathbb{Z}}/\operatorname{Ker}\varphi_{m,r}$. If we can determine orders of all $\operatorname{Ker}\varphi_{m,r}$ then we can get the order of $K_2(FG)$. By the facts in the proof in Theorem 3.1, we only need to deal with the case when G is a finite abelian p-group.

When R is a regular ring, essentially of finite type over a field of positive characteristic $p > 0$, by Theorem 2.5 in [3], $\operatorname{Ker}\varphi_{m,r}$ depends only on r and is the subgroup $D_{r,R}$ of $\Omega^1_{R/\mathbb{Z}}$ generated by $\{a^{p^l-1}da | 0 \le l < r, a \in R\}$. When R is not regular, for example $R = FG$, where F is a finite field of characteristic p and G is a finite abelian p-group, by the computations in Lemma 1.10 in [3], $\operatorname{Ker}\varphi_{m,r}$ still contains $D_{r,R}$ but does not coincide in general. Here is an example.

Example 2. Suppose $R = F_3[\mathbb{Z}_3 \times \mathbb{Z}_3]$, $\mathbb{Z}_3 \times \mathbb{Z}_3 = \langle\sigma\rangle \times \langle\tau\rangle$. Then by Theorem 2.2, $\Omega^1_{R/\mathbb{Z}}$ is the free R-module with basis $d\sigma, d\tau$. Put $m = 1, p = 3, r = 1$, we have

$$\Omega^1_{R/\mathbb{Z}} \xrightarrow{\varphi} K_2(R[t]/(t^4),(t)) \xrightarrow{f} K_2(R[t]/(t^4)).$$

By the definition $D_{1,R} = \langle a^{p^l-1}da | 0 \le l < 1\rangle = \langle da | a \in R\rangle$. An easy computation shows that $D_{1,R}$ is the F_3-space with basis $\{d\sigma, d\tau, \sigma d\sigma, \tau d\tau, \tau d\sigma + \sigma d\tau, 2\sigma\tau d\sigma + \sigma^2 d\tau, 2\sigma\tau d\tau + \tau^2 d\sigma, \tau\sigma^2 d\tau + \sigma\tau^2 d\sigma\}$. Obviously $\sigma^2 d\sigma, \tau^2 d\tau \notin D_{1,R}$. However

$$f \circ \varphi(\sigma^2 d\sigma) = f(\langle\sigma^2 t^3, \sigma\rangle) = \{1 - \sigma^3 t^3, \sigma\} = \{(1-t)^3, \sigma\} = \{1 - t, \sigma^3\} = 1.$$

Similarly, $f \circ \varphi(\tau^2 d\tau) = 1$. Since f is an injective map, $\varphi(\sigma^2 d\sigma) = \varphi(\tau^2 d\tau) = 1$. Then $\sigma^2 d\sigma, \tau^2 d\tau \in \operatorname{Ker}\varphi$. Hence $\operatorname{Ker}\varphi \supsetneqq D_{1,R}$.

Theorem 4.1. *Suppose $R = FG$, F is a finite field of odd characteristic p and G is a finite abelian group. Let $A = R[t]/(t^{mp^r+1})$ with $r \ge 1$, $(m, p) = 1$. If $\overline{a_1}, \dots, \overline{a_s}$ is the F_p-basis of G/G^p, A^s is the free A-module with generators $da_1, \dots, da_s$, then there is a map $K_2(A,(t)) \xrightarrow{f} \wedge^2_A(A^s)$.*

Proof. First we consider the test map

$$d\log : K_2(A,(t)) \to \wedge^2_A \Omega^1_{A/\mathbb{Z}}$$
$$\langle a, b\rangle \mapsto \frac{da \wedge db}{1 - ab}.$$

Let $D : A \to \Omega^1_{R/\mathbb{Z}} \otimes_R A$ be defined by applying the derivative $d : R \to \Omega^1_{R/\mathbb{Z}}$ to each coefficient, then according to Section 2 of [4], there is an A-module isomorphism

$$\Omega^1_{A/\mathbb{Z}} \approx \Omega^1_{A/R} \oplus (\Omega^1_{R/\mathbb{Z}} \otimes_R A)$$
$$df \mapsto \left(\frac{\partial f}{\partial t}dt, Df\right).$$

Then $\wedge^2_A \Omega^1_{A/\mathbb{Z}} \approx \wedge^2_A \Omega^1_{A/R} \oplus (\Omega^1_{A/R} \otimes_A (\Omega^1_{R/\mathbb{Z}} \otimes_R A)) \oplus \wedge^2_A(\Omega^1_{R/\mathbb{Z}} \otimes_R A)$, and by Theorem 2.2, $\Omega^1_{R/\mathbb{Z}}$ is a free R-module with generators $da_1, \dots, da_r$, so $\Omega^1_{R/\mathbb{Z}} \otimes_R A$ is a free A-modules with the same generators, now

$$K_2(A,(t)) \xrightarrow{d\log} \wedge^2_A \Omega^1_{A/\mathbb{Z}} \xrightarrow{\pi_3} \wedge^2_A(\Omega^1_{R/\mathbb{Z}} \otimes_R A) \xrightarrow{\psi} \wedge^2_A A^s.$$

The $f = \psi \circ \pi_3 \circ d\log$ is the map we want to obtain. $\square$

Let R be as above, $\wedge_R^* \Omega_{R/\mathbb{Z}}^1$ the algebra of differential forms over R, $(\wedge_R^* \Omega_{R/\mathbb{Z}}^1, d)$ the de Rham complex of R, and $H_{DR}^*(R)$ the de Rham cohomology of R.

Corollary 4.2. *Let A, R and f be as above, φ_{mr}: $\Omega_{R/\mathbb{Z}}^1 \to K_2(A, (t))$, then the cocycle Z^1 of the de Rham complex $(\wedge_R^* \Omega_{R/\mathbb{Z}}^1, d)$ is equal to the* $\mathrm{Ker}(f \circ \varphi_{m,r})$.

Proof. By the definition, $f \circ \varphi_{m,r}$ is the composite of the following maps

$$\Omega_{R/\mathbb{Z}}^1 \xrightarrow{\varphi_{m,r}} K_2(A, (t)) \xrightarrow{d\log} \wedge_A^2 \Omega_{A/\mathbb{Z}}^1 \xrightarrow{\pi_3} \wedge_A^2(\Omega_{R/\mathbb{Z}}^1 \otimes_R A) \xrightarrow{\psi} \wedge_A^2(A^s)$$

where $f \circ \varphi_{m,r}(adb) = \psi \circ \pi_3 \circ d\log(\langle at^{mp^r}, b\rangle) = \psi \circ \pi_3(\frac{d(at^{mp^r})\wedge db}{1-abt^{mp^r}}) = \psi(t^{mp^r}(da \otimes 1) \wedge (db \otimes 1)) = t^{mp^r} da \wedge db$. By Theorem 2.2, $\Omega_{R/\mathbb{Z}}^1$ is the free R-module with basis $da_1, \dots, da_s$, hence $\Omega_{R/\mathbb{Z}}^2$ is the free R-module with basis $da_i \wedge da_j$, $1 \le i < j \le s$, and $\wedge_A^2 A^s$ is a free A-module with the same generators. Define an R-homomorphism $g : \Omega_{R/\mathbb{Z}}^2 \to \wedge_A^2 A^s$ by $g(da \wedge db) = t^{mp^r} da \wedge db$, obviously g is injective. The $g \circ d^1$ is a homomorphism $\Omega_{R/\mathbb{Z}}^1 \to \wedge_A^2 A^s$ such that $g \circ d^1(adb) = t^{mp^r} da \wedge db = f \circ \varphi_{m,r}(adb)$. Hence $g \circ d^1 = f \circ \varphi_{m,r}$. Since g is an injective map, we have $\mathrm{Ker}(f \circ \varphi_{m,r}) = \mathrm{Ker}(g \circ d^1) = \mathrm{Ker}\, d^1 = Z^1$. $\square$

Theorem 4.3. *Suppose F is a finite field of odd prime characteristic p, G is a finite abelian p-group with cyclic decomposition $G = \langle x_1\rangle \times \cdots \times \langle x_n\rangle$, $\mathrm{ord}(x_i) = p^{l_i}$, $1 \le i \le n$. Let $G_i = \prod_{j\ne i}\langle x_j\rangle$. Then $H_{DR}^1(FG)$ is an F-space with basis $S = \{gx_i^j dx_i | 1 \le i \le n, 0 \le j < p^{l_i}, j \equiv -1 \bmod p, g \in G_i^p\}$.*

Proof. If $T = \sum f_{\alpha_1,\dots,\alpha_n} X_1^{\alpha_1} \cdots X_n^{\alpha_n} \in F[X_1, \dots, X_n]$, the polynomial ring in $X_1, \dots, X_n$ over F, the formal partial derivative $\frac{\partial T}{\partial X_i}$ is defined by

$$\frac{\partial T}{\partial X_i} = \sum \alpha_i f_{\alpha_1,\dots,\alpha_n} X_1^{\alpha_1} \cdots X_i^{\alpha_i - 1} \cdots X_n^{\alpha_n}.$$

If $x \in FG$, then $x = H(x_1, \dots, x_n)$ for some polynomial $H(X_1, \dots, X_n)$ in $F[X_1, \dots, X_n]$, thus $dx = \sum_{i=1}^n \frac{\partial H}{\partial X_i}(x_1, \dots, x_n)dx_i \in \Omega_{FG/\mathbb{Z}}^1$. By Theorem 2.2, $\Omega_{FG/\mathbb{Z}}^1$ is a free FG-module with basis $dx_1, \dots, dx_n$. If $v \in \Omega_{FG/\mathbb{Z}}^1$, then

$$v = \sum_{j=1}^n H_j(x_1, \dots, x_n)dx_j,$$

where $H_j(X_1, \dots, X_n) \in F[X_1, \dots, X_n]$, $j = 1, \dots, n$. Then

$$dv = \sum_{i<j} \left(\frac{\partial H_j}{\partial X_i} - \frac{\partial H_i}{\partial X_j}\right)(x_1, \dots, x_n)dx_i \wedge dx_j.$$

Since $\Omega_{FG/\mathbb{Z}}^2$ is the free FG-module with basis $dx_i \wedge dx_j$, $1 \le i < j \le n$, then $dv = 0$ if and only if

$$\frac{\partial H_j}{\partial X_i} = \frac{\partial H_i}{\partial X_j}, \quad 1 \le i < j \le n. \tag{4.1}$$

We can write $H_1(X_1, \dots, X_n)$ in the following

$$H_1(X_1, \dots, X_n) = \sum_{i=0}^{p^{l_1}-1} Q_i(X_2, \dots, X_n)X_1^i,$$

with $Q_i \in F[X_2, \dots, X_n]$. Let $w = \sum_{i\not\equiv -1 \bmod p}^{p^{l_1}-2} (i+1)^{-1} Q_i(x_2, \dots, x_n)x_1^{i+1}$. Then

$$v - dw = H_1'(x_1, \dots, x_n)dx_1 + \sum_{j=2}^n H_j'(x_1, \dots, x_n)dx_j,$$

where $H_1' = \sum_{i\equiv -1} Q_i$. Since $d(v - dw) = dv - ddw = 0$, then $v - dw \in Z^1$, by (4.1) we have

$$\frac{\partial H_1'}{\partial X_j} = \frac{\partial H_j'}{\partial X_1}, \quad 2 \le j \le n. \tag{4.2}$$

Suppose $H'_j(X_1, \ldots, X_n) = \sum_{i=1}^{p^{l_1}-1} Q_{ji}(X_2, \ldots, X_n)X_1^i,\ j \geq 2$ then by (4.2)

$$\sum_{i \equiv -1 \bmod p} \frac{\partial Q_i(X_2, \ldots, X_n)}{\partial X_j} X_1^i = \sum_{i=1}^{p^{l_1}-1} iQ_{ji}(X_2, \ldots, X_n)X_1^{i-1}.$$

By comparing the degree of X_1 of the two polynomials above, we conclude that both sides are equal to 0, so we have

$$Q_i(X_2, \ldots, X_n) = P_i(X_2^p, \ldots, X_n^p), \quad i \equiv -1 \bmod p$$
$$H'_j(X_1, \ldots, X_n) = P'_j(X_1^p, X_2, \ldots, X_n), \quad j = 2, \ldots, n.$$

Now we have found $v_1 \in Z^1$ with $\overline{v_1} = \bar{v}$ in $H^1_{DR}(FG)$, and

$$v_1 = \left(\sum_{i \equiv -1} P_i(x_2^p, \ldots, x_n^p)x_1^i\right) dx_1 + \sum_{j=2}^{n} P'_j(x_1^p, x_2, \ldots, x_n)dx_j.$$

Now using induction and repeating the above process we can eventually find $v_n \in Z^1$ such that $\overline{v_n} = \bar{v}$ in H^1_{DR} and

$$v_n = \sum_{i=1}^{n} \left(\sum_{j \equiv -1} T_{ij}(x_1^p, \ldots, \widehat{x_i^p}, \ldots, x_n^p)x_i^j\right) dx_i.$$

Obviously v_n can be generated by S, $S \subseteq Z^1$, $S \bigcap B^1 = \{0\}$, and S is an F-independent set since Ω^1_{FG} is an F-space with basis $\{gdx_i | g \in G, 1 \leq i \leq n\}$. So S is an F-basis of $H^1_{DR}(FG)$. □

Corollary 4.4. *Let F be as above, G an elementary abelian p-group with independent generators $x_1, \ldots, x_n$. Then $H^1_{DR}(FG)$ is an n-dimensional F-vector space with basis $\{x_i^{p-1}dx_i | 1 \leq i \leq n\}$.*

Proof. Since G is an elementary abelian p-group, $G_i^p = 1, i = 1, \ldots, n$. Now the conclusion follows immediately from Theorem 4.3. □

Proposition 4.5. *Let F and G be as in Theorem 4.3. Then the coboundary B^1 of $\left(\Omega^*_{FG/\mathbb{Z}}, d\right)$ has basis $S = \{dg | g \in G - G^p\}$ as an F-vector space.*

Proof. Suppose $x \in FG$, then $x = H(x_1, \ldots, x_n)$, where $H(X_1, \ldots, X_n)$ is a polynomial in $F[X_1, \ldots, X_n]$. Thus $dx = \sum_{i=1}^n \frac{\partial H}{\partial X_i}(x_1, \ldots, x_n)dx_i$. Hence $dx = 0$ if and only if $\frac{\partial H}{\partial X_i} = 0,\ 1 \leq i \leq n$. This implies $H(X_1, \ldots, X_n) = H_1(X_1^p, \ldots, X_n^p)$ for some polynomial $H_1(X_1, \ldots, X_n)$. If $g_1, \ldots, g_m \in G - G^p$, $\sum_{i=1}^m f_i dg_i = 0, f_i \in F$, that is $d\left(\sum_{i=1}^m f_i g_i\right) = 0$, so

$$\sum_{i=1}^{m} f_i g_i = H'(x_1^p, \ldots, x_n^p), \quad H' \in F[X_1, \ldots, X_n].$$

We conclude that $f_i = 0, 1 \leq i \leq m$, S is an F-independent set. Obviously B^1 is generated by S, now the proposition is proved. □

Theorem 4.6. *Suppose F is a finite field of odd prime characteristic p, G is an elementary abelian p-group with independent generators $g_1, \ldots, g_n$. Set $\varphi_{m,r}$: $\Omega^1_{FG/\mathbb{Z}} \to K_2\left(FG[t]/(t^{mp^r+1}), (t)\right), (m, p) = 1, r \geq 1$. Then $\operatorname{Ker}\varphi_{m,r}$ has basis $S = \{dg, g_i^{p-1}dg_i | g \in G - \{1\}, 1 \leq i \leq n\}$ as an F-vector space.*

Proof. By Theorem 4.3, $\operatorname{Ker}\varphi_{m,r} \subseteq \langle S\rangle$, where $\langle S\rangle$ is the F-vector space generated by S. By Lemma (1.10) in [3], $\{dg | g \in G - \{1\}\} \subseteq \operatorname{Ker}\varphi_{m,r}$. Since $f : K_2\left(FG[t]/(t^{mp^r+1}), (t)\right) \to K_2\left(FG[t]/(t^{mp^r+1})\right)$ is injective and

$$f \circ \varphi(g_i^{p-1}dg_i) = f(\langle g_i^{p-1}t^{mp^r}, g_i\rangle) = \{1 - t^{mp^r}, g_i\} = \{1 - t^{mp^{r-1}}, 1\} = 1,$$

it implies that $\langle S\rangle \subseteq \operatorname{Ker}\varphi_{m,r}$. Thus $\operatorname{Ker}\varphi_{m,r} = \langle S\rangle$. The independence of S follows from Proposition 4.5. □

Theorem 4.7. *Let F be a finite field of odd prime characteristic p, G is an arbitrary finite abelian p-group. Let $\tilde{S} = \{a^{p^l-1}da, g^{p^r-1}d_g | 0 \leq l < r, g \in G, g^{p^r} = 1\}$, then $\langle\tilde{S}\rangle \subseteq \operatorname{Ker}\varphi_{m,r}$.*

Proof. By Theorem 2.5 in [3], $D_{r,FG} = \langle a^{p^l-1}da | a \in FG, 0 \leq l < r\rangle \subseteq \operatorname{Ker}\varphi_{m,r}$. Since $f : K_2\left(FG[t]/(t^{mp^r+1}), (t)\right) \to K_2(FG[t]/(t^{mp^r+1}))$ is injective, and

$$\begin{aligned} f \circ \varphi_{m,r}(g^{p^r-1}dg) &= f(\langle g^{p^r-1}t^{mp^r}, g\rangle) \\ &= \{1 - g^{p^r}t^{mp^r}, g\} \\ &= \{(1-gt^m)^{p^r}, g\} \\ &= \{1-gt^m, g^{p^r}\} \\ &= \{1-gt^m, 1\} = 1. \end{aligned}$$

Thus $g^{p^r-1}dg \in \operatorname{Ker}\varphi_{m,r}$, so $\langle\tilde{S}\rangle \subseteq \operatorname{Ker}\varphi_{m,r}$. $\square$

Remark. For an arbitrary finite abelian p-group G, we guess that $\operatorname{Ker}\varphi_{m,r}$ is generated by $\{a^{p^l-1}da, g^{p^r-1}dg | a \in FG, 0 \leq l < r, g \in G, g^{p^r} = 1\}$. By Theorem 4.6, this is true when G is an elementary abelian p-group.

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References

[1] R.K. Dennis, M.E. Keating, M.R. Stein, Lower bounds for the order of $Wh_2(G)$, Math. Ann. 223 (1976) 97–103.
[2] B. Magurn, Explicit K_2 of some finite group rings, J. Pure Appl. Algebra 209 (2007) 801–911.
[3] J. Stienstra, On K_2 and K_3 of truncated polynomial rings, in: Algebraic K-Theory (Evanston, 1980), in: Lecture Notes in Math., vol. 854, Springer, Berlin, 1971.
[4] L. Roberts, S. Geller, K_2 of some truncated polynomial rings, in: Ring Theory Waterloo, in: Lecture Notes in Math., vol. 734, Springer Verlag, 1978.
[5] S. Bloch, Algebraic K-theory and crystalline cohomology, Publ. Math. IHES 47 (1977) 187–268.
[6] R.K. Dennis, M.R. Stein, K_2 of discrete valuation rings, Adv. Math. 18 (2) (1975) 182–238.
[7] R. Oliver, Lower bounds for $K_2^{\text{top}}\widehat{Z_p}\pi$ and $K_2(Z\pi)$, J. Algebra 94 (2) (1985) 425–487.

Further reading

[1] C.S. Seshadri, L'opération de Cariter. Applications. Exposé 6 Séminaire C. Chevalley, 4 (1958–1959).
[2] D. Husemoller, Cyclic Homology, Springer-Verlag, 1991.
[3] W. Van der Kallen, J. Stienstra, The relative K_2 of truncated polynomial rings, J. Pure Appl. Algebra 34 (2–2) (1984) 277–290.
[4] A.L. Borel, Linear Algebraic Groups, W.A. Benjamin, New York, 1969.